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LETTER TO THE EDITOR

On point transformations of evolution equations

J G Kingston
Department of Mathematics, University of Nottingham, Nottingham NG7 2RD, UK

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Abstract. Two evolution equations are considered of the form $u_t = H_i(x, t, u, u_x, u_{xx}, \ldots)$, $i = 1, 2$, which are related by the general point transformation $x^* = P(x, t, u)$, $t^* = Q(x, t, u)$, $u^* = R(x, t, u)$. It is shown that the time transformation must necessarily be of the form $t^* = Q(t)$ and that if H_i are polynomials in the spatial derivatives of u then $x^* = P(x, t)$.

Infinitesimal invariant point transformations of partial differential equations (PDEs) of the form

$$\begin{aligned} x^* &= x + \varepsilon X(x, t, u) + O(\varepsilon^2) \\ t^* &= t + \varepsilon T(x, t, u) + O(\varepsilon^2) \\ u^* &= u + \varepsilon U(x, t, u) + O(\varepsilon^2) \end{aligned} \tag{1}$$

relating independent variables x, t and dependent variable u with x^*, t^*, and u^* have been successfully applied to many areas of physics. Once such a transformation has been found for a PDE or a system of PDEs it implies the existence of a one-parameter (ε) Lie group of *finite* point transformations

$$x^*(\varepsilon) = P(x, t, u; \varepsilon) \qquad t^*(\varepsilon) = Q(x, t, u; \varepsilon) \qquad u^*(\varepsilon) = R(x, t, u; \varepsilon) \tag{2}$$

with the property that $x^*(0) = x$, $t^*(0) = t$, $u^*(0) = u$. As ε varies (x^*, t^*, u^*) traces out an *orbit* in $\mathbb{R}^3$ which passes through (x, t, u).

More generally there may also exist finite transformations of the form (2) which do not form a group and, in the absence of the parameter ε, the transformation (2) may not belong to any one-parameter group of transformations.

For infinitesimal transformations of form (1) the relations between partial derivatives can be defined systematically using a recursive procedure which is very clearly described by Olver [1]. In the particular case of an evolution equation of the form $H(x, t, u, u_t, u_x, u_{xx}, \ldots) = 0$ Tu [2] obtained explicit expressions for the transformed partial derivatives. Tu also proved that for evolution equations of this form the time transformation takes the simple form

$$t^* = t + \varepsilon Q(t) + O(\varepsilon^2)$$

the interesting feature being that Q is independent of both x and u. This is a striking result and has been exploited for example by Doyle and Englefield [3] who used the result to simplify the analysis of infinitesimal transformations of generalized Burger's equations relevant to problems with sound waves in ducts and shock waves.

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Here Tu's result is generalized to general finite point transformations which relate two evolution equations not necessarily of the same form. In addition it is shown that the spatial part of the transformation must be independent of u for a wide class of evolution equations. In the case of infinitesimal transformations these results make it easier to develop similarity solutions. This is due to the simplified form of the characteristic (Lagrange) equations associated with the first order PDE of the invariant surface (see for example Ames [4, p 125]).

Consider the two evolution equations

$$u_t = H_i(x, t, u, u_x, u_{xx}, \ldots) = 0 \qquad i = 1, 2. \tag{3}$$

Where H_i depends on the independent variables x and t and the dependent variable $u = u(x, t)$ and its derivatives with respect to x up to order n (≥ 2). Suppose that the equations (3), with one of them ($i = 1$) expressed in terms of x^*, t^*, u^* instead of x, t, u, are related by the point transformation

$$x^* = P(x, t, u) \qquad t^* = Q(x, t, u) \qquad u^* = R(x, t, u). \tag{4}$$

Assume that this is a non-degenerate transformation in that the Jacobian

$$J = \frac{\partial(P, Q, R)}{\partial(x, t, u)} \neq 0 \tag{5}$$

and also that

$$\delta = \frac{\partial(P(x, t, u(x, t)), Q(x, t, u(x, t))}{\partial(x, t)} \neq 0. \tag{6}$$

In (6) P and Q are regarded as functions of x and t as opposed to (5) where P, Q and R are regarded as functions of independent variables x, t, u.

For a function $\phi = \phi(x, t, u)$,

$$\mathrm{d}\phi = (\phi_X \ \ \phi_T)\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix} \tag{7}$$

where

$$\phi_X = \frac{\partial \phi}{\partial x} + u_x \frac{\partial \phi}{\partial u} \qquad \phi_T = \frac{\partial \phi}{\partial t} + u_t \frac{\partial \phi}{\partial u} \tag{8}$$

are the 'total derivatives' (see e.g. Olver [1]) of ϕ with respect to x and t respectively.

In particular, using $\phi = P$ and then $\phi = Q$,

$$\begin{pmatrix} \mathrm{d}x^* \\ \mathrm{d}t^* \end{pmatrix} = \begin{pmatrix} P_X & P_T \\ Q_X & Q_T \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix} \tag{9}$$

so that in general, for $\phi(x, t, u(x, t))$,

$$\mathrm{d}\phi = \frac{1}{\delta}(\phi_X \ \ \phi_T)\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} \mathrm{d}x^* \\ \mathrm{d}t^* \end{pmatrix} \tag{10}$$

giving the partial derivatives $\partial\phi/\partial x^*$ and $\partial\phi/\partial t^*$. Thus the partial derivatives $\partial u^*/\partial x^*$ and $\partial u^*/\partial t^*$ may be determined immediately in terms of x, t, u, t_t, u_x from the relation

$$\mathrm{d}u^* = \frac{1}{\delta}(R_X \ \ R_T)\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} \mathrm{d}x^* \\ \mathrm{d}t^* \end{pmatrix}. \tag{11}$$

For a function ψ of x, t, u and derivatives, possibly mixed, of u up to order n an expression similar to (10) can still be derived, that is

$$\mathrm{d}\psi = \frac{1}{\delta}(\psi_X \ \psi_T)\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} \mathrm{d}x^* \\ \mathrm{d}t^* \end{pmatrix} \tag{12}$$

where the total derivatives (8) must be extended to account for the derivatives of u which appear in ψ. That is, redefine the total derivatives of ψ with respect to x and t to be

$$\begin{aligned} \psi_X &= \frac{\partial\psi}{\partial x} + \sum_{i+j\leq n}\sum u_{i+1,j}\frac{\partial\psi}{\partial u_{i,j}} \\ \psi_T &= \frac{\partial\psi}{\partial t} + \sum_{i+j\leq n}\sum u_{i,j+1}\frac{\partial\psi}{\partial u_{i,j}} \end{aligned} \tag{13}$$

where

$$u_{i,j} = \frac{\partial^{i+j}u}{\partial x^i \partial t^j}. \tag{14}$$

This definition reduces to the simpler form (8) when ψ is a function of x, t and u only, such as when $\psi = P$, Q or R. The notation $u^*_{i,j}$ will refer to

$$\frac{\partial^{i+j}u^*}{\partial x^{*i}\partial t^{*j}}.$$

The procedure now is to write the first of equations (3) in terms of the starred variables, use the transformation (4) to express this equation in terms of x, t, u and the derivatives of u and then identify this with the equation $u_t = H_2$. The latter step may be achieved for example by using $u_t = H_2$ to eliminate u_t and then insisting that the remaining equation be an identity in the variables x, t, u and the remaining derivatives of u.

Theorem 1. For the point transformaion (4) relating the two evolution equations (3), $t^* = Q(t)$.

Lemma 1. If $u^*_{r,0}$ is expressed in terms of x, t, u and the x, t-derivatives of u then

$$\frac{\partial u^*_{r,0}}{\partial u_{0,r}} = (-1)^r \frac{JQ_X^r}{\delta^{r+1}} \qquad r \geq 1. \tag{15}$$

Proof is by induction on r:

$$\begin{aligned} \frac{\partial u^*_{r+1,0}}{\partial u_{0,r+1}} &= \frac{\partial}{\partial u_{0,r+1}}\left\{\frac{\partial}{\partial x^*}u^*_{r,0}\right\} \\ &= \frac{\partial}{\partial u_{0,r+1}}\left\{((u^*_{r,0})_X(u^*_{r,0})_T)\frac{1}{\delta}\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \end{aligned}$$

using (12) with $\psi = u^*_{r,0}$,

$$= \frac{1}{\delta}\begin{pmatrix} 0 & \dfrac{\partial u^*_{r,0}}{\partial u_{0,r}} \end{pmatrix}\begin{pmatrix} Q_T \\ -Q_X \end{pmatrix}$$

using (13) and noting that for $r \geq 1$ the term $u_{0,r+1}$ only appears in the second term of the row vector,

$$= (-1)^{r+1} \frac{JQ_X^{r+1}}{\delta^{r+2}}$$

from the induction hypothesis. For the basis of the induction consider firstly, from (11),

$$u^*_{x^*} = \frac{1}{\delta}(R_X \;\; R_T)\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{R_X Q_T - R_T Q_X}{P_X Q_T - Q_X P_T}. \tag{16}$$

Noting from (8) that

$$\frac{\partial \phi_T}{\partial u_t} = \phi_u \qquad \frac{\partial \phi_X}{\partial u_t} = 0$$

(16) may be differentiated to give

$$\frac{\partial u^*_{1,0}}{\partial u_{0,1}} \equiv \frac{\partial u^*_{x^*}}{\partial u_t} = -\frac{JQ_X}{\delta^2}$$

which is (15) with $r = 1$, completing the induction and proof of lemma 1.

Returning to the evolution equation

$$u^*_{t^*} = H_1(x^*, t^*, u^*, \ldots, u^*_{n,0})$$

firstly use the earlier results to express this equation in terms of x, t, $u_{i,j}$ where $u_{i,j}$ represents all cases such that $i, j \geq 0$, $i + j \leq n$. To identify this with the second evolution equation $u_t = H_2(x, t, u, \ldots, u_{n,0})$ use this latter equation to eliminate u_t and insist that the former equation is now an identity in the remaining variables x, t, $u_{i,j}$ where $i, j \geq 0$, $i + j \leq n$, exluding $(i, j) = (0, 1)$. In particular since the order of derivatives cannot increase under the point transformation the term $u_{0,n}$ can only arise from the term $u^*_{n,0}$ in H_1. Hence assuming that H_1 has explicit dependence on $u^*_{n,0}$ it must follow that

$$\frac{\partial u^*_{n,0}}{\partial u_{0,n}} = 0.$$

Lemma 1 now shows that $Q_X = Q_x + u_x Q_u$ must identically vanish for all x, t, u, u_x. Hence $Q_x = Q_u = 0$ and $Q = Q(t)$ as required.

With $Q = Q(t)$,

$$\delta = P_X Q_T - P_T Q_X = P_X Q_t \neq 0 \tag{17}$$

and

$$J = Q_t(P_x R_u - R_x P_u) \neq 0. \tag{18}$$

Equation (12) simplifies to

$$\mathrm{d}\psi = \frac{1}{P_X Q_t}(\psi_X \;\; \psi_T)\begin{pmatrix} Q_t & -P_T \\ 0 & P_X \end{pmatrix}\begin{pmatrix} \mathrm{d}x^* \\ \mathrm{d}t^* \end{pmatrix}$$

so that

$$\psi_{x^*} = \frac{1}{P_X}\psi_X \qquad \psi_{t^*} = -\frac{1}{P_X Q_t}(P_T \psi_X - P_X \psi_T). \tag{19}$$

In particular,

$$u^*_{t^*} = u^*_{0,1} = -\frac{1}{P_X Q_t}(P_T R_X - P_X R_T)$$

$$= -\frac{1}{P_X Q_t}\{(R_x P_t - R_t P_x) + u_x(R_u P_t - R_t P_u) + u_t(R_x P_u - R_u P_x)\} \qquad (20)$$

$$u^*_{x^*} = u^*_{1,0} = \frac{R_X}{P_X} = \left(\frac{1}{P_X} D\right) R$$

denoting R_X by DR,

$$u^*_{x^* x^*} = u^*_{2,0} = \left(\frac{1}{P_x} D\right)^2 R \qquad u^*_{n,0} = \left(\frac{1}{P_x} D\right)^n R \qquad n \geq 1. \qquad (21)$$

Substituting for $u_t (= H_2(x, t, u, \ldots, u_{n,0}))$ in the transformed form of $u^*_{t^*} = H_1(x^*, t^*, u^*, \ldots, u^*_{n,0})$ gives

$$-\frac{1}{P_X Q_t}\{(R_x P_t - R_t P_x) + u_x(R_u P_t - R_t P_u) + (R_x P_u - R_u P_x) H_2(x, t, u, \ldots, u_{n,0})\}$$

$$\equiv H_1\left(P, Q, R, \left(\frac{1}{P_X} D\right) R, \ldots, \left(\frac{1}{P_X} D\right)^n R\right). \qquad (22)$$

Equation (22) must be an identity in the variables $x, t, u, u_x, \ldots, u_{n,0}$.

Theorem 2. If the point transformation (4) relates the two evolution equations (3) in which H_i, $i = 1, 2$, are polynomials in the derivatives of u, then $x^* = P(x, t)$.

Note that H_1 and H_2 do not need to be polynomials in u itself.

The proof below is untidy in expression but simple in method which is, roughly speaking, to find the coefficient of the term in equation (22) which contains the highest powers of the highest-order deriatives. This coefficient is found to contain non-zero factors with the exception of P_u which must therefore vanish. The following lemma will be needed.

Lemma 2. If $u^*_{r,0}$ is expressed in terms of x, t, u and the x, t-derivatives of u then

$$\frac{\partial u^*_{r,0}}{\partial u_{r,0}} = \begin{cases} R_u / P_X & r = 1 \\ J/(P_X^{r+1} Q_t) & r \geq 2. \end{cases} \qquad (23)$$

The proof of this lemma is by induction on r.

$$\frac{\partial u^*_{r+1,0}}{\partial u_{r+1,0}} = \frac{\partial}{\partial u_{r+1,0}}\left\{\frac{\partial}{\partial x^*} u^*_{r,0}\right\}$$

$$= \frac{\partial}{\partial u_{r+1,0}}\left\{\frac{1}{P_X}(u^*_{r,0})_X\right\} \qquad \text{from (19)}$$

$$= \frac{1}{P_X}\frac{\partial u^*_{r,0}}{\partial u_{r,0}} \qquad \text{from (13), } r \geq 1$$

$$= \frac{J}{P_X^{r+2} Q_t} \qquad r \geq 2$$

from the induction hypothesis. This leaves cases $r = 1$ and $r = 2$ to verify in (23), which is readily done using (20) and (21) and then (18), confirming lemma 2.

Suppose that the leading term in $H_2(x, t, u, \ldots, u_{n,0})$ is

$$f_2(x, t, u)u_{n,0}^{\alpha_n}u_{n-1,0}^{\alpha_{n-1}}\ldots u_{1,0}^{\alpha_1} \tag{24}$$

where $f_2(x, t, u)\neq 0$, $n\geqslant 2$, $\alpha_n\geqslant 1$ is the highest power of the highest-order derivative, $\alpha_{n-1}\geqslant 0$ is the highest power of $u_{n-1,0}$ in the coefficient of $u_{n,0}^{\alpha_n}$, and so on, with $\alpha_1\geqslant 0$. Similarly the leading term in $H_1(x^*, t^*, u^*, \ldots, u_{n,0}^*)$ will be of the form

$$f_1(x^*, t^*, u^*)u_{n,0}^{*\beta_n}u_{n-1,0}^{*\beta_{n-1}}\ldots u_{1,0}^{*\beta_1} \tag{25}$$

where $f_1(x^*, t^*, u^*)\neq 0$, $\beta_n\geqslant 1$.

Retaining the leading term on both the LHS and the RHS of (22) and making use of (24), (25), lemma 2 and (18), produces the two terms

$$\frac{1}{P_X Q_t}\left(\frac{J}{Q_t}\right)f_2(x, t, u)u_{n,0}^{\alpha_n}\ldots u_{1,0}^{\alpha_1} \qquad f_1(P, Q, R)\frac{1}{P_X^a}\left(\frac{J}{Q_t}\right)^b u_{n,0}^{\beta_n}\ldots u_{1,0}^{\beta_1}R_u^{\beta_1}$$

where

$$a=(n+1)\beta_n+n\beta_{n-1}+\ldots+3\beta_2+\beta_1$$

$$b=\beta_n+\beta_{n-1}+\ldots+\beta_2.$$

Note that $a\geqslant 3$.

Multiplying by P_X^a and again retaining only the leading terms gives

$$u_x^{a-1}\frac{P_u^{a-1}}{Q_t}\left(\frac{J}{Q_t}\right)f_2(x, t, u)u_{n,0}^{\alpha_n}\ldots u_{1,0}^{\alpha_1} \qquad f_1(P, Q, R)\left(\frac{J}{Q_t}\right)^b u_{n,0}^{\beta_n}\ldots u_{1,0}^{\beta_1}R_u^{\beta_1}. \tag{26}$$

The right-hand side term is non-zero and must be matched by the left-hand side term unless that term vanishes (which can only happen when $P_u=0$). The next or subsequent term would then have to match the right-hand side term. For the existing terms in (26) to match it is necessary for $\beta_i=\alpha_i$, $i=2,\ldots,n$ and for $a-1+\alpha_1-\beta_1=0$. Inspection of a above and remembering that all α_i, β_i are non-negative and that $a\geqslant 3$ shows that the latter condition is impossible to achieve. Hence $P_u=0$.

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